

Upper bounds on the Q-spectral radius of book-free and/or $K_{s,t}$ -free graphs *

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Abstract

In this paper, we prove two results about the signless Laplacian spectral radius $q(G)$ of a graph G of order n with maximum degree Δ . Let $B_n = K_2 + \overline{K_n}$ denote a book, i.e., the graph B_n consists of n triangles sharing an edge.

(1) Let $1 < k \leq l < \Delta < n$ and G be a connected $\{B_{k+1}, K_{2,l+1}\}$ -free graph of order n with maximum degree Δ . Then

$$q(G) \leq \frac{1}{4}[3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta + n - 1)}].$$

with equality holds if and only if G is a strongly regular graph with parameters (Δ, k, l) .

(2) Let $s \geq t \geq 3$, and let G be a connected $K_{s,t}$ -free graph of order n ($n \geq s+t$). Then

$$q(G) \leq n + (s - t + 1)^{1/t} n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.$$

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1 Introduction

Our graph notation follows Bollobás [1]. In particular, let $G = (V(G), E(G))$ be a simple graph. Denote by $v(G)$ the order of G and $e(G)$ the size of G , that is to say, $v(G) = |V(G)|$, and $e(G) = |E(G)|$. Set $\Gamma_G(u) = \{v | uv \in E(G)\}$, and $d_G(u) = |\Gamma_G(u)|$, or simply $\Gamma(u)$ and $d(u)$, respectively. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote the minimal degree and maximal degree of graph G , respectively.

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For a simple graph G of order n , let the matrix $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$, and $A(G) = (a_{ij})_{n \times n}$ be the adjacency matrix of G with $a_{ij} = 1$ if v_i is adjacent to v_j , and $a_{ij} = 0$ otherwise. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G . The largest eigenvalue of $A(G)$ and $Q(G)$ are called spectral radius and signless Laplacian spectral radius (or simply Q-spectral radius) of G , respectively, and marked $\rho(G)$ and $q(G)$, respectively.

Let X be a set of vertices of G , $G[X]$ is the graph induced by X , and $e(X) = e(G[X])$. Let P_k , C_k and K_k be the path, cycle, and complete graph of order k , respectively. If all vertices of G have the same degree k , then G is k -regular. A k -regular graph is called *strongly regular* with parameters (k, a, c) whenever each pair of adjacent vertices have $a \geq 0$ common neighbors, and each pair of non-adjacent vertices have $c \geq 1$ common neighbors.

The main results of this paper are in the spirit of the trend in the famous Zarankiewicz problem [5]:

Problem A *How many edges can have a graph of order n if it does not contain a complete bipartite subgraph $K_{s,t}$?*

In 1996, Füredi [4] gave an upper bound on the above Zarankiewicz problem. In 2010, Nikiforov [6] improved his result. That is, if G is a $K_{s,t}$ -free graph of order n , then

$$e(G) \leq \frac{1}{2}(s-t+1)^{1/t}n^{2-1/t} + \frac{1}{2}(t-1)n^{2-2/t} + \frac{1}{2}(t-2)n.$$

The spectral version of the Zarankiewicz problem is the following one:

Problem B *How large can be the spectral radius $\rho(G)$ of a graph G of order n that does not contain $K_{s,t}$?*

There are some results for some value of s and t .

In 2007, the upper bound on the signless Laplacian spectral radius of $K_{2,l+1}$ -free graph as the corollary of the following Lemma 1.1 was proved in [9] by Shi and Song.

Lemma 1.1. *$0 \leq k \leq l \leq \Delta < n$ and G be a connected $\{B_{k+1}, K_{2,l+1}\}$ -free graph of order n with maximum degree Δ . Then*

$$\rho(G) \leq [k-l + \sqrt{(k-l)^2 + 4\Delta + 4l(n-l)}]/2,$$

with equality if and only if G is a strongly regular with parameters (Δ, k, l) .

In 2007, Nikiforov [7] improved the above bound showing that

Lemma 1.2. *Let $l \geq k \geq 0$. If G is a $\{B_{k+1}, K_{2,l+1}\}$ -free graph of order n with maximum degree Δ . Then*

$$\rho(G) \leq \min\{\Delta, \frac{1}{2}[k-1 + 1 + \sqrt{(k-l+1)^2 + 4l(n-1)}]\}.$$

If G is connected, equality holds if and only if one of the following conditions holds:

- (1) $\Delta^2 - \Delta(k-l+1) \leq l(n-1)$ and G is Δ -regular;
- (2) $\Delta^2 - \Delta(k-l+1) > l(n-1)$ and every two vertices of G have k common neighbors if they are adjacent, and l common neighbors otherwise.

Setting $l = \Delta$ or $k = l$, Lemma 1.2 implies assertions that strengthen Corollaries 1 and 2 of [9].

In 2010, Nikiforov [6] also gave a bound as the following lemma.

Lemma 1.3. *let $s \geq t \geq 2$, and let G be a $K_{s,t}$ -free graph of order n . If $t = 2$, then*

$$\rho(G) \leq 1/2 + \sqrt{(s-1)(n-1) + 1/4}.$$

If $t \geq 3$, then

$$\rho(G) \leq (s-t+1)^{1/t} n^{1-1/t} + (t-1)n^{1-2/t} + t-2.$$

and

$$e(G) < \frac{1}{2}(s-t+1)^{1/t} n^{2-1/t} + \frac{1}{2}(t-1)n^{2-2/t} + \frac{1}{2}(t-2)n.$$

A newer trend in extremal graph theory is the Zarankiewicz problem for signless Laplacian spectral radius of graphs:

Problem C *How large can be the signless Laplacian spectral radius $q(G)$ of a graph G of order n that does not contain subgraph $K_{s,t}$?*

When $s = t = 2$, we notice that the $K_{2,2}$ -free graph is the same as C_4 -free graph. Also in 2013, de Freitas [2] has proved that if G contains no C_4 , then

$$q(G) < q(F_n),$$

unless $G = F_n$, where F_n is the friendship graph of order n . For n odd, F_n is a union of $\lfloor n/2 \rfloor$ triangles sharing a single common vertex, and for n even, F_n is obtained by hanging an edge to the common vertex of F_{n-1} .

In this paper, we discuss upper bounds on the signless Laplacian spectral radius of Book-free and/or $K_{2,l+1}$ -free ($l > 1$) graphs of order n with maximum degree Δ .

Theorem 1.4. *Let $1 < k \leq l < \Delta < n$ and G be a connected $\{B_{k+1}, K_{2,l+1}\}$ -free graph of order n with maximum degree Δ . Then*

$$q(G) \leq \frac{1}{4}[3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta + n - 1)}]. \quad (1)$$

with equality holds if and only if G is a strongly regular graph with parameters (Δ, k, l) .

Because every graph is obviously $K_{2,\Delta+1}$ -free, Theorem 1.4 readily implies a sharp upper bound for book-free graph.

Corollary 1.5. *Let $1 < k < \Delta < n$ and G be a connected B_{k+1} -free graph of order n with maximum degree Δ . Then*

$$q(G) \leq \frac{1}{4}[\Delta + k + 1 + \sqrt{(\Delta + k + 1)^2 + 32\Delta(n-1)}].$$

with equality if and only if G is a strongly regular graph with parameters (Δ, k, Δ) .

Becase a $K_{2,l}$ -free graph is also B_l -free. Theorem 1.4 with $k = l$ also implies a sharp upper bound for $K_{2,l}$ -free graphs.

Corollary 1.6. *Let $1 < l < \Delta$ and G be a connected $K_{2,l+1}$ -free graph of order n with maximum degree Δ . Then*

$$q(G) \leq \frac{1}{4}[3\Delta - l + 1 + \sqrt{(3\Delta - l + 1)^2 + 32l(n-1)}].$$

with equality if and only if G is a strongly regular graph with parameters (Δ, l, l) .

Furthermore we will discuss $s \geq t \geq 3$, let G be a connected graph of order n , when $n < s + t$, then G must contain no $K_{s,t}$, so we only discuss $n \geq s + t$.

Theorem 1.7. *Let $s \geq t \geq 3$, and let G be a connected $K_{s,t}$ -free graph of order n ($n \geq s + t$). Then*

$$q(G) \leq n + (s - t + 1)^{1/t} n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.$$

2 Main Lemmas

In this section, we state some well-know results which will be used in this paper.

Lemma 2.1. *Let $s \geq 2$, $t \geq 2$, $0 \leq k \leq s - 2$, and let $G(A, B)$ be a bipartite graph with parts A and B . Suppose that G contains no copy of $K_{s,t}$ with a vertex class of size s in A and a vertex class of size t in B . Then $G(A, B)$ has at most*

$$(s - k - 1)^{1/t} |B| |A|^{1-1/t} + (t - 1) |A|^{1+k/t} + k |B|$$

edges.

Lemma 2.2. *([3], [8]) For every graph G , we have*

$$q(G) \leq \max_{u \in V(G)} \left\{ d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v) \right\}.$$

3 Proofs

Proof of Theorem 1.4. Let Q_i denote the i th row vector of $Q (= Q(G))$ and let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be the Perron-eigenvector of Q corresponding to $q(G)$. Then $x_i > 0$ for $1 \leq i \leq n$. Since G is $\{B_{k+1}, K_{2,l+1}\}$ -free, each pair of adjacent vertices has at most k common neighbors and each pair of non-adjacent vertices has at most l common neighbors. Thus

$$\sum_{i=1}^n \sum_{v_p, v_q \in \Gamma(v_i)} x_p x_q \leq k \sum_{v_p v_q \in E(G)} x_p x_q + l \sum_{v_p v_q \notin E(G)} x_p x_q. \quad (2)$$

Then by virtue of $\mathbf{x}^T A(K_n) \mathbf{x} \leq \rho(K_n) = n - 1$. Thus

$$\begin{aligned} q(G) &= \mathbf{x}^T Q \mathbf{x} = \mathbf{x}^T D \mathbf{x} + \mathbf{x}^T A \mathbf{x} = \sum_{i=1}^n d_i x_i^2 + 2 \sum_{v_i v_p \in E(G)} x_i x_p \\ &\leq \Delta + \mathbf{x}^T A(K_n) \mathbf{x} - 2 \sum_{v_i v_p \notin E(G)} x_i x_p \\ &\leq \Delta + n - 1 - 2 \sum_{v_i v_p \notin E(G)} x_i x_p. \end{aligned}$$

Also we can obtain

$$\begin{aligned}
q(G) &= \mathbf{x}^T Q \mathbf{x} = \sum_{i=1}^n \sum_{j=1, i < j}^n 2q_{i,j} x_i x_j + \sum_{i=1}^n d_i x_i^2 \\
&\leq \sum_{i=1}^n \sum_{j=1, i < j}^n q_{i,j} (x_i^2 + x_j^2) + \sum_{i=1}^n d_i x_i^2 \\
&= \sum_{i=1}^n \sum_{j=1, i < j}^n q_{i,j} x_i^2 + \sum_{i=1}^n d_i x_i^2 \\
&= 2 \sum_{i=1}^n d_i x_i^2.
\end{aligned}$$

So

$$\sum_{i=1}^n d_i x_i^2 \geq \frac{q}{2}.$$

Then

$$\begin{aligned}
q^2(G) &= \|Q\mathbf{x}\|^2 = \sum_{i=1}^n (Q_i \mathbf{x})^2 = \sum_{i=1}^n (d_i x_i + \sum_{v_i v_p \in E(G)} x_p)^2 \\
&= \sum_{i=1}^n [d_i^2 x_i^2 + 2d_i x_i \sum_{v_i v_p \in E(G)} x_p + (\sum_{v_i v_p \in E(G)} x_p)^2] \\
&= \sum_{i=1}^n d_i^2 x_i^2 + 2 \sum_{i=1}^n d_i \sum_{v_i v_p \in E(G)} x_i x_p + \sum_{i=1}^n d_i x_i^2 + 2 \sum_{i=1}^n \sum_{v_p, v_q \in \Gamma(v_i)} x_p x_q \\
&\leq (\Delta + 1) \sum_{i=1}^n d_i x_i^2 + 2\Delta \sum_{i=1}^n \sum_{v_i v_p \in E(G)} x_i x_p \\
&\quad + 2k \sum_{v_p v_q \in E(G)} x_p x_q + 2l \sum_{v_p v_q \notin E(G)} x_p x_q \tag{3} \\
&= (\Delta + 1) \sum_{i=1}^n d_i x_i^2 + (4\Delta + 2k) \sum_{v_i v_p \in E(G)} x_i x_p + 2l \sum_{v_p v_q \notin E(G)} x_p x_q \\
&\leq (2\Delta + k) \left(\sum_{i=1}^n d_i x_i^2 + 2 \sum_{v_i v_p \in E(G)} x_i x_p \right) \\
&\quad - (\Delta + k - 1) \sum_{i=1}^n d_i x_i^2 + 2l \sum_{v_p v_q \notin E(G)} x_p x_q \\
&\leq (2\Delta + k)q - \frac{\Delta + k - 1}{2}q + l(\Delta + n - 1 - q) \\
&= \frac{1}{2}(3\Delta + k - 2l + 1)q + l(\Delta + n - 1).
\end{aligned}$$

Solving the inequality gives the upper bound

$$q(G) \leq \frac{1}{4} [3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta + n - 1)}].$$

If the upper bound of (1) is attained then all inequalities in the above argument must be equalities. In particular, from (2) and $x_i > 0$ for $1 \leq i \leq n$, we have that each pair of adjacent vertices in G has exactly k common neighbors and each pair of non-adjacent vertices in G has exactly l common neighbors. Moreover, by (3), G must be Δ -regular. Thus G must be a strongly regular graph with parameters (Δ, k, l) . \square

Proof of Theorem 1.7. By Lemma 2.2, let w be a vertex of G such that

$$d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) = \max_{u \in V(G)} \left\{ d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v) \right\}.$$

Then

$$q(G) \leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i).$$

Note first that if $d(w) \leq s + t - 1$, then

$$\begin{aligned} q(G) &\leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) \leq d(w) + \Delta(G) \\ &\leq s + t - 1 + n - 1 = s + t + n - 2 \\ &\leq n + (s - t + 1)^{1/t} n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3. \end{aligned}$$

Therefore we shall assume that $s + t - 1 \leq d(w) \leq n - 1$. Let U and W be disjoint sets satisfying $|U| = d(w)$ and $|W| = n - 1$, and let φ_U and φ_W be bijections

$$\varphi_U : U \rightarrow \Gamma(w), \varphi_W : W \rightarrow V(G) \setminus \{w\}.$$

Define a bipartite graph H with vertex classes U and W by joining $u \in U$ and $v \in W$ whenever $\{\varphi_U(u), \varphi_W(v)\} \in E(G)$.

Then we can get that H does not contain a copy of $K_{s-1,t}$ with $s - 1$ vertices in W and t vertices in U . Indeed, the map $\psi : V(H) \rightarrow V(G)$ defined as

$$\psi(x) = \begin{cases} \varphi_U(x), & \text{if } x \in U, \\ \varphi_W(x), & \text{if } x \in W. \end{cases}$$

is a homomorphism of H into $G - w$. Assume for a contradiction that $F \subset H$ is a copy of $K_{s-1,t}$ with a set of S of $s - 1$ vertices in W and a set of T of t vertices in U . Clearly S and T are the vertex classes of F . Note that $\psi(F)$ is a copy of $K_{s-1,t}$ in $G - w$, and $\psi(S) = \varphi_W(S) \subset V(G) \setminus \{w\}$ and $\psi(T) = \varphi_U(T) \subset \Gamma_G(w)$ are the vertex classes of $\psi(F)$ of size $s - 1$ and size t , respectively. Now, adding w to $\psi(F)$, we see that G contains a $K_{s,t}$, a contradiction proving the claim.

Suppose that $0 \leq k \leq \min\{s, t\} - 2$. Setting $k' = k - 1, s' = s - 1, t' = t, A = W, B = U$, then from Lemma 2.1, we have

$$\begin{aligned} e(H) &\leq (s - k - 1)^{1/t} |U| |W|^{1-1/t} + (k - 1) |U| + (t - 1) |W|^{1+(k-1)/t} \\ &= (s - k - 1)^{1/t} d(w) n^{1-1/t} + (k - 1) d(w) + (t - 1) (n - 1)^{1+(k-1)/t}. \end{aligned}$$

On the other hand. We have

$$e(H) = \sum_{v \in \Gamma(w)} d(v) - d(w),$$

and so,

$$\sum_{v \in \Gamma(w)} d(v) \leq ((s - k - 1)^{1/t} n^{1-1/t} + k)d(w) + (t - 1)(n - 1)^{1+(k-1)/t}.$$

And then from Lemma 2.2, we have

$$\begin{aligned} q(G) &\leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) \\ &\leq d(w) + \frac{(t - 1)(n - 1)^{1+(k-1)/t}}{d(w)} + (s - k - 1)^{1/t} n^{1-1/t} + k. \end{aligned}$$

Since the function

$$f(x) = x + \frac{(t - 1)(n - 1)^{1+(k-1)/t}}{x}$$

is convex for $x > 0$, its maximum in any closed interval is attained at one of the ends of this interval. In the case $s + t - 1 \leq d(w) \leq n - 1$, then,

$$\begin{aligned} q(G) &\leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) \\ &\leq \max\left\{s + t - 1 + \frac{(t - 1)(n - 1)^{1+(k-1)/t}}{s + t - 1}, n - 1 + \frac{(t - 1)(n - 1)^{1+(k-1)/t}}{n - 1}\right\} \\ &\quad + (s - k - 1)^{1/t} n^{1-1/t} + k \\ &\leq (s - k - 1)^{1/t} n^{1-1/t} + k + \frac{(t - 1)(n - 1)^{1+(k-1)/t}}{n - 1} + n - 1 \\ &= (s - k - 1)^{1/t} n^{1-1/t} + k + (t - 1)(n - 1)^{(k-1)/t} + n - 1. \end{aligned}$$

Now, if $s \geq t \geq 3$, setting $k = t - 2$, we obtain

$$q(G) \leq n + (s - t + 1)^{1/t} n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.$$

So, the proof is complete. \square

References

- [1] B. Bollobás, Modern Graph Theory, *Graduate Texts in Mathematics*, Springer-Verlag, New York, xiv+394 pp, (1998).
- [2] M.A.A. de Freitas, V. Nikiforov, and L. Patuzzi, Maxima of the Q-index: forbidden 4-cycle and 5-cycle, *Electron. J. Linear Algebra* **26** (2013) 905-916.
- [3] L.H. Feng and G.H. Yu, On three conjectures involving the signless laplacian spectral radius of graphs, *Publ. Inst. Math. (Beograd) (N.S.)*, **85** (2009), 35-38.
- [4] Z. Füredi, An upper bound on Zarankiewicz's problem, *Comb. Probab. Comput* **5**(1996), 29-33.
- [5] V. Nikiforov, Some new results in extremal graph theory, *Surveys in Combinatorics*, Cambridge: Cambridge University Press (2011), 141-181.

- [6] V. Nikiforov, A contribution to the Zarankiewicz problem, *Linear Algebra Appl.* **432**(2010), 1405-1411.
- [7] V. Nikiforov, Bounds on graph eigenvalues II, *Linear Algebra Appl.* **427**(2007)183-189.
- [8] R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra Appl.* **295**(1998), 33-35.
- [9] L.S. Shi and Z.P. Song, Upper bounds on the spectral radius of book-free and/or $K_{2,l}$ -free graphs, *Linear Algebra Appl.* **420**(2007), 526-529.